THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 3: Sequences

1. Use the definition of the limit of a sequence to establish the following limits.

   (a) $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$,

   (b) $\lim_{n\to\infty} \frac{2n}{n+1} = 2$,

   (c) $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = 2$.

2. Let $\alpha \in \mathbb{R}$. Prove that $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$.

3. Use the definition of the limit of a sequence to show that the following sequences diverge.

   (a) $a_n = (-1)^n$

   (b) $a_{2n} = 0$ and $a_{2n-1} = n$ for $n \in \mathbb{N}$

4. Give an example of two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n - y_n$ exists, but both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ do not exist.

5. Give an example of two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n y_n$ exists, but both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ do not exist.

6. Prove that $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$. Give an example to show that the convergence of $\{|x_n|\}$ need not imply the convergence of $\{x_n\}$.

7. Prove that if $\lim_{n\to\infty} x_n = x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

8. (a) Suppose that $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. Prove that $x \geq 0$. Give an example for $x = 0$.

   (b) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $a_n > b_n$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n$.

9. Suppose that $\{x_n\}$ is convergent sequence and $\{y_n\}$ is a sequence such that for any $\epsilon > 0$ there exists $M > 0$ such that $|x_n - y_n| < \epsilon$ for all $n \geq M$. Does it follow that $\{y_n\}$ convergent?

10. (The convergence of Cesaro averages) Suppose that the sequence $\{a_n\}$ converges to a. Define the sequence $\{x_n\}$ by
$$x_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$
for every natural number n. Prove that $\lim_{n\to 0} x_n = a$.

11. Suppose that the sequence $\{a_n\}$ converges to a and that $|a| < 1$. Prove that the sequence $\{(a_n)^n\}$ converges to 0.

12. (a) Let $\{x_n\}$ be a sequence of real numbers. Show that $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} x_n^2 = 0$.

    (b) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $\lim_{n\to\infty} a_n^2 + b_n^2 = 0$. Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

13. Define $a_1 = 1$ and for all $n \in \mathbb{N}$, define $a_{n+1} = \dfrac{1+a_n}{2+a_n}$. Prove that $\{a_n\}$ converges and find the limit.

14. Define $x_1 = 8$ and for all $n \in \mathbb{N}$, define $x_{n+1} = \dfrac{x_n}{2} + 2$. Prove that $\{a_n\}$ converges and find the limit.

15. Prove that the sequence
$$\{1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\}$$
converges.

16. Let $\{b_n\}$ be a bounded sequence of nonnegative real numbers and r be a real number such that $0 \le r < 1$. Define
$$s_n = b_1 r + b_2 r^2 + \cdots + b_n r^n$$
for every natural number n. Prove that $\{s_n\}$ converges.

17. Prove that $\{\sin n\}$ is divergent.

18. Give an example to show that the Bolzano-Weierstrass Theorem fails if boundedness assumption is dropped.

19. A set of real numbers K is said to be compact provided that every sequence in K has a subsequence that converges to a point in K. Suppose A is a subset of $\mathbb{R}$. Prove that A is compact if and only if A is closed and bounded.
(Remark: By the Bolzano-Weierstrass Theorem, $[a, b]$ is a compact subset of $\mathbb{R}$.)

20. Suppose that $x_n \ge 0$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} (-1)^n x_n$ exists. Show that $\{x_n\}$ converges.

21. Show directly from the definition that the following are Cauchy sequences.

    (a) $\dfrac{n+1}{n}$;

    (b) $1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$.

22. Show directly that a bounded, increasing sequence is a Cauchy sequence.

23. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that $\{x_n\}$ is a Cauchy sequence.

24. A sequence $\{x_n\}$ is said to be **contractive** if there exists a constant C, $0 < C < 1$, such that
$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$$
for all $n \in \mathbb{N}$. Prove that every contractive sequence is a Cauchy sequence, and therefore is convergent.

25. For any $x \in \mathbb{R}$ and define $f_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$. Prove that for any fixed x, $\{f_n(x)\}$ is a Cauchy sequence, and therefore is convergent.

(Remark: If $f : \mathbb{R} \to \mathbb{R}$ is a function defined by $f(x) = \lim_{n \to \infty} f_n(x)$, then $f(x)$ in fact equals to e^x. We may use the similar idea to define $\cos x$ and $\sin x$.)